Technical Comments

Comments on "A Review of Microrocket Technology: 10^{-6} to 1 lbf Thrust"

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Nomenclature

ΔI = impulse bit per thruster, per firing, lb-sec
J = vehicle moment of inertia, slug-ft^2
θ = angular position, rad
F = thrust level of each thruster, lbf
r = moment arm of each thruster, ft
I_{sp} = specific impulse, sec
$\dot{w}$ = propellant flowrate per thruster, lb/sec
$\bar{\dot{w}}$ = average total propellant consumption, lb/sec
W = total propellant consumption per cycle, lb
$\Delta\dot{\theta}_0$ = total angular rate change per firing, rad/sec
$\dot{\theta}_1$ = initial angular rate, rad/sec
$\dot{\theta}_2$ = angular rate after first firing, rad/sec
t_0 = firing time (pulse width), sec
θ_r = one-half the limit cycle deadband, rad

SUTHERLAND and Maes[1] pointed out in their article that minimum impulse bit (ΔI) has a strong influence on the limit-cycle efficiency and that halving ΔI can be twice as effective as doubling specific impulse (I_{sp}). While this is essentially true, it should be remembered that the impulse bit is the product of an average thruster on-time, propellant flowrate and the effective I_{sp}. Thus, the $(\Delta I)^2/I_{sp}$ portion of the average propellant consumption equation can be reduced to the form $(\dot{w})^2(t_0)^2I_{sp}$, where t_0 is the average on-time and $\dot{w}$ the propellant flowrate per thruster. Now if the only requirement is to minimize propellant consumption, the ideal solution is to minimize the thruster on-time, the propellant flowrate, and the specific impulse. On the other hand, the derivation of this equation is based on the assumption that ΔI is fixed at some level based on maximum disturbing torques, secondary propulsion system requirements, and guidance system limitations.[2] The original form of the equation must therefore be used and specific impulse should be as high as possible.

Although the form of the average propellant consumption equation is generally agreed upon, the numerical coefficient changes each time the equation appears. I feel that the exact equation should be documented as it is being used in the design of various vehicles. The most probable average propellant consumption equations are as follows:

$$\bar{\dot{w}} = \frac{2}{3}\frac{(\Delta I)^2 r}{\theta_r J I_{sp}} \quad \text{(for a pair of coupled thrusters)} \quad (1)$$

$$\bar{\dot{w}} = \frac{1}{6}\frac{(\Delta I)^2 r}{\theta_r J I_{sp}} \quad \text{(for a pair of thrusters)} \quad (2)$$

The derivation of these equations is presented in the following paragraphs together with the maximum possible propellant consumption equations. The numerical coefficient varies because the actual propellant consumption is a function of a statistically distributed oscillation frequency.

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The total angular rate change per firing for a pair of coupled thrusters is given by

$$\Delta\dot{\theta}_0 = 2I_{sp}\dot{w}rt_0/J \quad (3)$$

The equivalent total propellant consumption per cycle is therefore

$$W = 4\dot{w}t_0 = 2\Delta\dot{\theta}_0 J/I_{sp}r \quad (4)$$

The average period per cycle (t_1) is given by

$$t_1 = 2\theta_r/|\dot{\theta}_1| + 2\theta_r/|\dot{\theta}_2| \quad (5)$$

where $|\Delta\dot{\theta}_0|$ is equal to:

$$|\Delta\dot{\theta}_0| = |\dot{\theta}_1| + |\dot{\theta}_2| \quad (6)$$

Equation (5) can therefore be combined to give

$$t_1 = \frac{2\theta_r|\Delta\dot{\theta}_0|}{|\dot{\theta}_1|[|\Delta\dot{\theta}_0| - |\dot{\theta}_1|]} \quad (7)$$

The average total propellant consumption is therefore

$$\bar{\dot{w}} = (2\Delta\dot{\theta}_0 J/I_{sp}r)(1/t_1) \quad (8)$$

The frequency (f_1) of the limit cycle oscillation is

$$f_1 = \frac{1}{t_1} = \frac{\dot{\theta}_1\,\Delta\dot{\theta}_0 - (\dot{\theta}_1)^2}{2\theta_r\,\Delta\dot{\theta}_0} \quad (9)$$

The initial angular rate ($\dot{\theta}_1$) at which the vehicle enters the deadband must be uniformly distributed between 0 and $\Delta\dot{\theta}_0$ to insure positive control. Therefore,

$$p(\dot{\theta}_1) = 1/\Delta\dot{\theta}_0 \quad (10)$$

The probability density function of the frequency is

$$p(f_1) = p(\dot{\theta}_1)|d\dot{\theta}_1/df_1| \quad (11)$$

where

$$df_1/d\dot{\theta}_1 = (\Delta\dot{\theta}_0 - 2\dot{\theta}_1)/[2\theta_r][\Delta\dot{\theta}_0] \quad (12)$$

Substituting (12) into (11) gives

$$p(f_1) = (1/\Delta\dot{\theta}_0)[2\theta_r\,\Delta\dot{\theta}_0/(\Delta\dot{\theta}_0 - 2\dot{\theta}_1)] = 2\theta_r/(\Delta\dot{\theta}_0 - 2\dot{\theta}_1) \quad (13)$$

The statistical mean of f_1 is:

$$\bar{f}_1 = \int_0^{\Delta\dot{\theta}_0} f_1\,p(f_1)df_1 = \int_0^{\Delta\dot{\theta}_0} \frac{\dot{\theta}_1\,\Delta\dot{\theta}_0 - (\dot{\theta}_1)^2}{2\theta_r(\Delta\dot{\theta}_0)^2}\,d\dot{\theta}_1 = \left(\frac{\Delta\dot{\theta}_0}{12\theta_r}\right) \quad (14)$$

The conditions for a maximum limit-cycle frequency can be obtained from Eq. (12):

$$\dot{\theta}_1|_{\text{max frequency}} = \tfrac{1}{2}\Delta\dot{\theta}_0 \quad (15)$$

The corresponding maximum frequency is

$$f_1|_{\max} = \Delta\dot{\theta}_0/8\theta_r \quad (16)$$

The corresponding oscillation periods for average and maximum propellant consumption are given in Eqs. (17) and (18), respectively:

$$t_1|_{\text{most probable}} = 12\theta_r/\Delta\dot{\theta}_0 \quad (17)$$

$$t_1|_{\max} = 8\theta_r/\Delta\dot{\theta}_0 \quad (18)$$

Table 1 Limit cycle flowrate numerical coefficients

	Most probable	Maximum
Pair of single thrusters	$\frac{1}{6}$	$\frac{1}{4}$
Pair of coupled thrusters	$\frac{2}{3}$	1

The total average propellant consumption for these two cases is determined by substituting in Eq. (8):

$$\bar{w}|_{\text{most probable}} = \tfrac{2}{3}(\Delta I)^2 r / J\theta_r I_{sp} \tag{19}$$

and

$$\bar{w}|_{\max} = (\Delta I)^2 r / J\theta_r I_{sp} \tag{20}$$

If only single thrusters are used instead of a pair of coupled thrusters to provide control, the average propellant consumption is reduced by a factor of 4. This is because the oscillation frequency is reduced by a factor of 2 as is the total propellant expended per firing. The corresponding flowrate relationships are given in Eqs. (21) and (22):

$$\bar{w}|_{\text{most probable}} = \tfrac{1}{6}(\Delta I)^2 r / J\theta_r I_{sp} \tag{21}$$

$$\bar{w}|_{\max} = \tfrac{1}{4}(\Delta I)^2 r / J\theta_r I_{sp} \tag{22}$$

The numerical coefficients for these four cases are summarized in Table 1.

References

[1] Sutherland, G. S. and Maes, M. E., "A review of microrocket technology: 10^{-6} to 1 lbf thrust," J. Spacecraft Rockets **3**, 1153–1165 (1966).

[2] Reeves, D. F., Boardman, W. P., and Baumann, H. A., "Pulsed rocket control techniques," ARS Paper 2704-62 (November 1962).

Comments on "Application of Biot's Variational Method to Convective Heating of a Slab"

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IN the improved approximation, wherein the change of q_1 caused by variation of θ_1 will be taken into account, Eqs. (13) and (14) and the expression for Q_1, combined to give Eq. (15) do not seem to completely account for q_1 being included in θ_1. Even though the final results in the approximate cases come out to be the same here, the final equation analogous to (15) is different. Proceeding from first principles:

$$Q_1 = \theta_1 \left(\frac{\partial H}{\partial q_1}\right)_{x=0} = \frac{c_v \theta_1^2}{3}\left\{1 + \frac{2k}{(uq_1 + 2k)}\right\}$$

D and V are the same as before. Upon differentiating,

$$\frac{\partial V}{\partial q_1} = c_v \theta_1^2 \left\{\frac{1}{10} + \frac{2}{5}\left[\frac{k}{(uq_1 + 2k)}\right]\right\}$$

$$\frac{\partial D}{\partial \dot{q}_1} = c_v^2 \theta_1^2 \left\{\frac{13}{315k} + \frac{4}{63}\frac{k}{(uq_1 + 2k)^2} + \frac{2}{21}\frac{1}{(uq_1 + 2k)}\right\} q_1 \dot{q}_1$$

Combining these, the Lagrangian equation including the change in q_1 caused by variation of θ_1 [analogous to Eq. (15)] becomes

$$\left\{\frac{13}{105} + \frac{1}{21}\left[\frac{4k^2}{(2k + uq_1)^2}\right] + \frac{1}{7}\left[\frac{2k}{(2k + uq_1)}\right]\right\} q_1 \dot{q}_1 = \frac{k}{c_v}\left\{\frac{7}{10} + \frac{2}{5}\left[\frac{2k}{(2k + uq_1)}\right]\right\}$$

and not Eq. (15) as given in Ref. 1.

Reference

[1] Chu, H. N., "Application of Biot's variational method to convective heating of a slab," J. Spacecraft Rockets **1**, 686–687 (1964).

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Reply by Author to C. L. Gupta's Comment

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THERE is an alternative way of deriving the expressions in the preceding comments, the Lagrangian multiplier method. Introducing the multiplier λ, one obtains three equations relating q_1, θ_1, and λ. In this way one need not decide whether to work with q_1 or θ_1, a priori. Either the preceding method or the Lagrangian multiplier method is mandatory, if the problem is regarded as one of purely mathematical exercise after the introduction of Eq. (4).

A third alternative is to do away with Eq. (1) and therefore regard θ_1 and q_1 as two independent generalized coordinates. In place of Eq. (14) one would then have an equation derived by variation with respect to θ_1. This is perfectly agreeable since Eq. (1) is simply a statement of Fourier's law, which may be approximated in the Biot variational scheme. This alternative also represents a mathematically correct method.

All these alternatives are more cumbersome than the one presented in my paper, whereas they improve the numerical results negligibly. For example, using Gupta's expressions, the result $q_1 = 2.646(kt/c_v)^{1/2}$ is obtained, instead of my $q_1 = 2.66(kt/c_v)^{1/2}$, a difference of less than 1%. In my paper, recognition is made of the fact that although Fourier's law may be approximated it does not have to be approximated. Since the effect of the convective boundary condition is the primary concern of the paper, Eq. (1) is adopted with the qualification that θ_1 is not regarded as a generalized coordinate but as a given function of t. Physically one realizes that q_1 is a rather arbitrarily defined quantity but θ_1 is not. Once Eq. (13) is obtained, the variational stage is past and any given function will enable one to obtain a solution of $q_1(t)$. Equation (4) fills this role and at the same time describes what happens near the boundary precisely. So Eq. (4) is adopted. In the problem of my paper, engineers are, more often than not, interested in accurately determining θ_1 rather than q_1. The method of my paper enables the determination of θ_1, less sensitively affected than q_1 is by a change in the assumed temperature profile from the quadratic to the cubic. I had the Lagrangian multiplier scheme in mind when I worked on the paper, but it was discarded along with some other encumbrances in favor of the simplified scheme presented in my paper.

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